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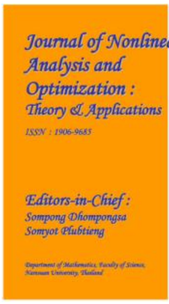

WEAK CONTRACTION CONDITION INVOLVING $d(x, y)$UNDER SIX SELF MAPPINGS IN METRIC SPACE

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Abstract.
In this paper, first we introduce generalized $\emptyset$ −weak contraction condition that involves quadratic terms of distance function $d(x, y)$and proved common fixed point theorems using weakly compatible for six self mappings. At the last we give corollaries and example in support of our theorem.

Keywords and phrases:
$\emptyset$ −Weak Contraction, Weakly Compatible mappings, Metric space.
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1. Introduction

The Banach fixed point theorem is a fundamental method for studying fixed point theory, demonstrating the presence and uniqueness of a fixed point under some conditions. This theorem establishes a method for resolving a wide range of applied problems in mathematics and engineering. The majority of applied mathematics problems reduce to inequality, whose solutions give rise to the f ixed point. It was the huge change of the fixed point theory literature when commutativity mappings was used by Jungck [3] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. The first ever attempt to relax the commutativity to weak commutativity was initiated by Sessa [9]. Further, in 1986 Jungck [4] introduced more generalized commutativity, so called compatibility. One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence.

2. Preliminaries

Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point.

Let $(\mathfrak{A}, d)$ be a complete metric space. If $\mathcal{A}: \mathfrak{A} \to \mathfrak{A}$ satisfies $d\big(\mathcal{A}(x), \mathcal{A}(y)\big) \leq k\big(d(x, y)\big)$, for all $x, y \in \mathfrak{A}$, $0 \leq k < 1$, then it has a unique fixed point.

In 1969, Boyd and Wong [2] replaced the constant k in Banach contraction principle by a control function ψ as follows:

Let $(\mathfrak{A}, d)$ be a complete metric space and $\psi: [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$.

If $\mathcal{A}: \mathfrak{A} \to \mathfrak{A}$ satisfies $d(\mathcal{A}(x), \mathcal{A}(y)) \leq \psi(d(x, y))$ for all $x, y \in \mathfrak{A}$, then it has a unique fixed point.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction as follows:

A map $\mathcal{A} : \mathfrak{A} \to \mathfrak{A}$ is said to be weak contraction if for each $x, y \in \mathfrak{A}$ there exists a function $\emptyset : [0, \infty) \to [0, \infty)$, $\emptyset(t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that

$$d(\mathcal{A}x, \mathcal{A}y) \leq d(x,y) - \emptyset(d(x,y))$$

Several fixed point theorems and common fixed point theorems have proved in the fixed point theory literature usingweak contraction as an implicit function.

In 1996, Jungck [6] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

Definition 2.1[6] Two self-mappings $\mathcal{A}$ and $\mathcal{B}$ on a metric space $(\mathfrak{A}, d)$are called weakly compatible if they commute at their coincidence point i.e.,if $\mathcal{A}u = \mathcal{B}u$, for some $u \in \mathfrak{A}$ then $\mathcal{AB}u = \mathcal{BA}u$.

2. Main Result

In 2013, Murthy and Prasad [8] proposed a new form of inequality for a map involving cubic terms of the metric function $d(x,y)$, which expanded and generalized the results of Alber and Gueree-Delabriere [1] and many others cited in the fixed point theory literature.

In 2013, Murthy and Prasad [8] proved the following result

Theorem 2.2.1 Let $\mathcal{T}$ be a map of a complete metric space$(\mathfrak{A}, d)$ into itself satisfying the following:

$$[1 + \hbar d(x,y)]d^2(\mathcal{T}x, \mathcal{T}y) \leq \hbar \max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(x,\mathcal{T}x)d(y,\mathcal{T}y) \\ +d(x,\mathcal{T}x)d^2(y,\mathcal{T}y) \end{bmatrix}, \\ d(x,\mathcal{T}x)d(x,\mathcal{T}y)d(y,\mathcal{T}x), \\ d(x,\mathcal{T}y)d(y,\mathcal{T}x)d(y,\mathcal{T}y) \end{matrix} \right\} + m(x,y) - \emptyset(m(x,y)),$$

where $$m(x,y) = \max \left\{ \begin{matrix} d^2(x,y), \\ d(x,\mathcal{T}x)d(y,\mathcal{T}y), \\ d(x,\mathcal{T}y)d(y,\mathcal{T}x), \\ \frac{1}{2}\begin{bmatrix} d(x,\mathcal{T}x)d(x,\mathcal{T}y) + \\ d(y,\mathcal{T}x)d(y,\mathcal{T}y) \end{bmatrix} \end{matrix} \right\},$$

$\hbar \geq 0$ is a real number and $\emptyset: [0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.Then $\mathcal{T}$has a unique fixed point in$\mathfrak{A}$.

The result of Murthy and Prasad [8] for a pair of weakly compatible mappings satisfying $\emptyset$ −weak contractive condition is extended and generalized in this section involving various combinations of metric functions in six self mappings.

Theorem 2.1 Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{P}$ and $\mathcal{Q}$ be six self-mappings of metric space $(\mathfrak{A}, d)$ satisfying the following conditions:

(C_1) $\mathcal{P}(\mathfrak{A}) \subseteq \mathcal{ST}(\mathfrak{A}), \mathcal{Q}(\mathfrak{A}) \subseteq \mathcal{AB}(\mathfrak{A})$,

(C_2) $\mathcal{AB} = \mathcal{BA}, \mathcal{ST} = \mathcal{TS}, \mathcal{PB} = \mathcal{BP}, \mathcal{QT} = \mathcal{TQ}$,

(C_3) One of $\mathcal{ST}(\mathfrak{A}), \mathcal{Q}(\mathfrak{A}), \mathcal{AB}(\mathfrak{A})or\mathcal{P}(\mathfrak{A})$is complete,

(C_4) The pairs $(\mathcal{P}, \mathcal{AB})$ and $(\mathcal{Q}, \mathcal{ST})$ are weakly compatible,

(C_5) $[1 + \hbar d(\mathcal{AB}x, \mathcal{ST}y)]d^2(\mathcal{P}x, \mathcal{Q}y)$

$$\leq \hbar \max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{ST}y,\mathcal{Q}y) \\ +d^2(\mathcal{ST}y,\mathcal{Q}y)d(\mathcal{AB}x,\mathcal{P}x) \end{bmatrix}, \\ d(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{AB}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{ST}y), \\ d(\mathcal{AB}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{ST}y)d(\mathcal{ST}y,\mathcal{Q}y) \end{matrix} \right\} + \sigma(\mathcal{AB}x, \mathcal{ST}y) - \emptyset\{\sigma(\mathcal{AB}x, \mathcal{ST}y)\},$$

where$$\sigma(\mathcal{AB}x, \mathcal{ST}y) = \max \left\{ \begin{matrix} d^2(\mathcal{AB}x,\mathcal{ST}y), \\ d(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{ST}y,\mathcal{Q}y), \\ d(\mathcal{AB}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{ST}y), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{AB}x,\mathcal{Q}y) \\ +d(\mathcal{P}x,\mathcal{ST}y)d(\mathcal{ST}y,\mathcal{Q}y) \end{bmatrix} \end{matrix} \right\},$$

for all $x, y \in \mathfrak{A}$, $\hbar \geq 0$ is a real number and $\emptyset: [0, \infty) \to [0, \infty)$ is a continuous function with

$\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > t$, for each $t > 0$. Then $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathfrak{A}$.

Proof. Let $x_0 \in \mathfrak{A}$ be an arbitrary point. From (C_1) we can find a point x_1 and x_2 such that $\mathcal{P}(x_0) = \mathcal{ST}(x_1)$ and $\mathcal{Q}(x_1) = \mathcal{AB}(x_2)$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $\mathfrak{A}$ such that

$y_{2n} = \mathcal{P}(x_{2n}) = \mathcal{ST}(x_{2n+1})$ and

$y_{2n+1} = \mathcal{Q}(x_{2n+1}) = \mathcal{AB}(x_{2n+2})$, for each $n \geq 0$.(2.1)

For brevity, we write $r_{2n} = d(y_{2n}, y_{2n+1})$

First, we prove that $\{r_{2n}\}$ is non-increasing sequence and converges to zero.

Case I If n is even, putting $x = x_{2n}$ and $y = x_{2n+1}$ in (C_5), we get

$$[1 + h d(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1})] d^2(\mathcal{P}x_{2n}, \mathcal{Q}x_{2n+1})$$

$$\leq h\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \\ + d^2(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) \end{bmatrix}, \\ d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) d(\mathcal{AB}x_{2n}, \mathcal{Q}x_{2n+1}) d(\mathcal{P}x_{2n}, \mathcal{ST}x_{2n+1}), \\ d(\mathcal{AB}x_{2n}, \mathcal{Q}x_{2n+1}) d(\mathcal{P}x_{2n}, \mathcal{ST}x_{2n+1}) d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \end{Bmatrix} +$$

$$\sigma(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1}) - \emptyset\{\sigma(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1})\},$$

where $\sigma(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1}) = max \begin{Bmatrix} d^2(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1}), \\ d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}), \\ d(\mathcal{AB}x_{2n}, \mathcal{Q}x_{2n+1}) d(\mathcal{P}x_{2n}, \mathcal{ST}x_{2n+1}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) d(\mathcal{AB}x_{2n}, \mathcal{Q}x_{2n+1}) \\ + d(\mathcal{P}x_{2n}, \mathcal{ST}x_{2n+1}) d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \end{bmatrix} \end{Bmatrix}$.

Using (2.1), we have

$$[1 + h d(y_{2n-1}, y_{2n})] d^2(y_{2n}, y_{2n+1})$$

$$\leq h\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1}) \\ + d^2(y_{2n}, y_{2n+1})\, d(y_{2n-1}, y_{2n}) \end{bmatrix}, \\ d(y_{2n-1}, y_{2n}) d(y_{2n-1}, y_{2n+1}) d(y_{2n}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1}) d(y_{2n}, y_{2n}) d(y_{2n}, y_{2n+1}) \end{Bmatrix}$$

$$+ \sigma(y_{2n-1}, y_{2n}) - \emptyset\big(\sigma(y_{2n-1}, y_{2n})\big),$$

where $\sigma(y_{2n-1}, y_{2n}) = max \begin{Bmatrix} d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}) d(y_{2n}, y_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(y_{2n-1}, y_{2n}) d(y_{2n-1}, y_{2n+1}) \\ + d(y_{2n}, y_{2n}) d(y_{2n}, y_{2n+1}) \end{bmatrix} \end{Bmatrix}$.

On using $r_{2n} = d(y_{2n}, y_{2n+1})$ in the above inequality, we have

$$[1 + h r_{2n-1}] r_{2n}^2 \leq h max \left\{\frac{1}{2}[r_{2n-1}^2 r_{2n} + r_{2n}^2 r_{2n-1}], 0, 0\right\}$$

$$+ \sigma(y_{2n-1}, y_{2n}) - \emptyset\big(\sigma(y_{2n-1}, y_{2n})\big),$$

where $\sigma(y_{2n-1}, y_{2n}) = max\left\{r_{2n-1}^2, r_{2n-1} r_{2n}, 0, \frac{1}{2}[r_{2n-1} d(y_{2n-1}, y_{2n+1}) + 0]\right\}$.

By using triangular inequality and property of $\emptyset$, we get

$d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) = r_{2n-1} + r_{2n}$

and $\sigma(y_{2n-1}, y_{2n}) \leq \sigma'(x, y) = max\left\{r_{2n-1}^2, r_{2n-1} r_{2n}, 0, \frac{1}{2}[r_{2n-1}(r_{2n-1} + r_{2n}), 0]\right\}$.

If $r_{2n-1} < r_{2n}$, then $h r_{2n}^2 \leq h r_{2n}^2 - \emptyset\{r_{2n}^2\}$, a contradiction.

Therefore, $r_{2n}^2 \leq r_{2n-1}^2$ implies $r_{2n} \leq r_{2n-1}$.

In a similar way, if n is odd, then we can obtain $r_{2n+1} < r_{2n}$.

It follows that the sequence $\{r_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} r_{2n} = m$, for some $m \geq 0$.

Suppose $m > 0$; then from inequality (C_5), we have

$$[1 + h d(\mathcal{AB}x_{2n}, \mathcal{ST}x_{2n+1})] d^2(\mathcal{P}x_{2n}, \mathcal{Q}x_{2n+1})$$

$$\leq \hbar \max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}d^2(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\\ +d^2(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})\end{array}\right],\\ d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{AB}x_{2n},\mathcal{Q}x_{2n+1})d(\mathcal{P}x_{2n},\mathcal{ST}x_{2n+1}),\\ d(\mathcal{AB}x_{2n},\mathcal{Q}x_{2n+1})d(\mathcal{P}x_{2n},\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\end{array}\right\} +$$

$$\sigma(\mathcal{AB}x_{2n},\mathcal{ST}x_{2n+1}) - \emptyset\{\sigma(\mathcal{AB}x_{2n},\mathcal{ST}x_{2n+1})\},$$

where $\sigma(\mathcal{AB}x_{2n},\mathcal{ST}x_{2n+1}) = \max\left\{\begin{array}{c}d^2(\mathcal{AB}x_{2n},\mathcal{ST}x_{2n+1}),\\ d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1}),\\ d(\mathcal{AB}x_{2n},\mathcal{Q}x_{2n+1})d(\mathcal{P}x_{2n},\mathcal{ST}x_{2n+1}),\\ \frac{1}{2}\left[\begin{array}{c}d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{AB}x_{2n},\mathcal{Q}x_{2n+1})\\ +d(\mathcal{P}x_{2n},\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\end{array}\right]\end{array}\right\}$.

Now by using triangular inequality and property of $\emptyset$ and taking limit $n \to \infty$, we get $[1+\hbar m]m^2 \leq \hbar m^3 + m^2 - \emptyset(m^2)$. Then $\emptyset(m^2) \leq 0$, since m is positive, then by property of $\emptyset$, we get $m = 0$. Therefore, we conclude that $\lim_{n\to\infty} r_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = m = 0$.

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$.

$d(y_{m(k)}, y_{n(k)}) \geq \epsilon,\ \ d(y_{m(k)}, y_{n(k)-1}) < \epsilon$

Now $\epsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$

Letting $k \to \infty$, we get $\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = \epsilon$

Now from the triangular inequality, we have,

$$|d(y_{n(k)}, y_{m(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{m(k)}, y_{m(k)+1}).$$

Taking limits as $k \to \infty$ we have $\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)+1}) = \epsilon$.

On using triangular inequality, we have

$$|d(y_{m(k)}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{n(k)}, y_{n(k)+1}).$$

Proceeding limits as $k \to \infty$ we get $\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)+1}) = \epsilon$.

Similarly, we have

$$|d(y_{m(k)+1}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{m(k)}, y_{m(k)+1}) + d(y_{n(k)}, y_{n(k)+1}).$$

Taking limit as $k \to \infty$ in the above inequality, we have $\lim_{k\to\infty} d(y_{n(k)+1}, y_{m(k)+1}) = \epsilon$.

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (C_5), we get

$$[1 + \hbar d(\mathcal{AB}x_{m(k)}, \mathcal{ST}x_{n(k)})]d^2(\mathcal{P}x_{m(k)}, \mathcal{Q}x_{n(k)})$$

$$\leq$$

$$\hbar \max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}d^2(\mathcal{AB}x_{m(k)},\mathcal{P}x_{m(k)})d(\mathcal{ST}x_{n(k)},\mathcal{Q}x_{n(k)})\\ +d^2(\mathcal{ST}x_{n(k)},\mathcal{Q}x_{n(k)})d(\mathcal{AB}x_{m(k)},\mathcal{P}x_{m(k)})\end{array}\right],\\ d(\mathcal{AB}x_{m(k)},\mathcal{P}x_{m(k)})d(\mathcal{AB}x_{m(k)},\mathcal{Q}x_{n(k)})d(\mathcal{P}x_{m(k)},\mathcal{ST}x_{n(k)}),\\ d(\mathcal{AB}x_{m(k)},\mathcal{Q}x_{n(k)})d(\mathcal{P}x_{m(k)},\mathcal{ST}x_{n(k)})d(\mathcal{ST}x_{n(k)},\mathcal{Q}x_{n(k)})\end{array}\right\} +$$

$$\sigma(\mathcal{AB}x_{m(k)},\mathcal{ST}x_{n(k)}) - \emptyset\{\sigma(\mathcal{AB}x_{m(k)},\mathcal{ST}x_{n(k)})\},$$

where $\sigma(\mathcal{AB}x_{m(k)},\mathcal{ST}x_{n(k)}) = \max\left\{\begin{array}{c}d^2(\mathcal{AB}x_{m(k)},\mathcal{ST}x_{n(k)}),\\ d(\mathcal{AB}x_{m(k)},\mathcal{P}x_{m(k)})d(\mathcal{ST}x_{n(k)},\mathcal{Q}x_{n(k)}),\\ d(\mathcal{AB}x_{m(k)},\mathcal{Q}x_{n(k)})d(\mathcal{P}x_{m(k)},\mathcal{ST}x_{n(k)}),\\ \frac{1}{2}\left[\begin{array}{c}d(\mathcal{AB}x_{m(k)},\mathcal{P}x_{m(k)})d(\mathcal{AB}x_{m(k)},\mathcal{Q}x_{n(k)})\\ +d(\mathcal{P}x_{m(k)},\mathcal{ST}x_{n(k)})d(\mathcal{ST}x_{n(k)},\mathcal{Q}x_{n(k)})\end{array}\right]\end{array}\right\}$,

Using (2.5) we obtain

$$[1+\hbar d(y_{m(k)-1},y_{n(k)-1})]d^2(y_{m(k)},y_{n(k)}) \le \hbar \max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}d^2(y_{m(k)-1},y_{m(k)})d(y_{n(k)-1},y_{n(k)})\\+d(y_{m(k)-1},y_{m(k)})d^2(y_{n(k)-1},y_{n(k)})\end{array}\right],\\ d(y_{m(k)-1},y_{m(k)})d(y_{m(k)-1},y_{n(k)})d(y_{m(k)},y_{n(k)-1}),\\ d(y_{m(k)-1},y_{n(k)})d(y_{m(k)},y_{n(k)-1})d(y_{n(k)-1},y_{n(k)})\end{array}\right\}$$

$$+\sigma(y_{m(k)-1},y_{n(k)-1})-\emptyset\{\sigma(y_{m(k)-1},y_{n(k)-1})\},$$

where $$\sigma(y_{m(k)-1},y_{n(k)-1}) = \max\left\{\begin{array}{c}d^2(y_{m(k)-1},y_{n(k)-1}),\\ d(y_{m(k)-1},y_{m(k)})d(y_{n(k)-1},y_{n(k)}),\\ d(y_{m(k)-1},y_{n(k)})d(y_{m(k)},y_{n(k)-1}),\\ \frac{1}{2}\left[\begin{array}{c}d(y_{m(k)-1},y_{m(k)})d(y_{m(k)-1},y_{n(k)})\\+d(y_{m(k)},y_{n(k)-1})d(y_{n(k)-1},y_{n(k)})\end{array}\right]\end{array}\right\}.$$

Letting $k\to\infty$, we get$[1+\hbar\epsilon]\epsilon^2 \le \hbar\max\left\{\frac{1}{2}[0+0],0,0\right\}+\epsilon^2-\emptyset(\epsilon^2)$ $=\epsilon^2-\emptyset(\epsilon^2)$, a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in $\mathfrak{A}$.

Case 1. $\mathcal{ST}(\mathfrak{A})$ is complete**.** In this case $\{y_{2n}\}=\{\mathcal{ST}x_{2n+1}\}$ is a Cauchy sequence in $\mathcal{ST}(\mathfrak{A})$, which is complete then the sequence $\{y_{2n}\}$ converges to a some point $z\in\mathcal{ST}(\mathfrak{A})$. Consequently, the subsequence's $\{Qx_{2n+1}\},\{\mathcal{AB}x_{2n}\},\{\mathcal{ST}x_{2n+1}\}$ and $\{\mathcal{P}x_{2n}\}$ also converges to the same point z. As $z\in\mathcal{ST}(\mathfrak{A})$, there exists $u\in\mathfrak{A}$ such that $z=\mathcal{ST}u$.

Now we claim that $z=Qu$. For this putting$x=x_{2n}$ and $y=u$ in (C_5), we get

$$[1+\hbar d(\mathcal{AB}x_{2n},\mathcal{ST}u)]d^2(\mathcal{P}x_{2n},Qu) \le \hbar\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}d^2(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{ST}u,Qu)\\+d^2(\mathcal{ST}u,Qu)d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})\end{array}\right],\\ d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{AB}x_{2n},Qu)d(\mathcal{P}x_{2n},\mathcal{ST}u),\\ d(\mathcal{AB}x_{2n},Qu)d(\mathcal{P}x_{2n},\mathcal{ST}u)d(\mathcal{ST}u,Qu)\end{array}\right\}$$

$+\sigma(\mathcal{AB}x_{2n},\mathcal{ST}u)-\emptyset\{\sigma(\mathcal{AB}x_{2n},\mathcal{ST}u)\}$,where $\sigma(\mathcal{AB}x_{2n},\mathcal{ST}u)=$

$$\max\left\{\begin{array}{c}d^2(\mathcal{AB}x_{2n},\mathcal{ST}u),\\ d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{ST}u,Qu),\\ d(\mathcal{AB}x_{2n},Qu)d(\mathcal{P}x_{2n},\mathcal{ST}u),\\ \frac{1}{2}\left[\begin{array}{c}d(\mathcal{AB}x_{2n},\mathcal{P}x_{2n})d(\mathcal{AB}x_{2n},Qu)\\+d(\mathcal{P}x_{2n},\mathcal{ST}u)d(\mathcal{ST}u,Qu)\end{array}\right]\end{array}\right\}.$$

Taking limit as $n\to\infty$, we have

$$[1+\hbar d(z,z)]d^2(z,Qu) \le \hbar\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}d^2(z,z)d(z,Qu)\\+d^2(z,Qu)d(z,z)\end{array}\right],\\ d(z,z)d(z,Qu)d(z,z),\\ d(z,Qu)d(z,z)d(z,Qu)\end{array}\right\}$$

$+\sigma(z,z)-\emptyset\{\sigma(z,z)\}$,

where$$\sigma(z,z)=\max\left\{\begin{array}{c}d^2(z,z),\\ d(z,z)d(z,Qu),\\ d(z,Qu)d(z,z),\\ \frac{1}{2}\left[\begin{array}{c}d(z,z)d(z,Qu)\\+d(z,z)d(z,Qu)\end{array}\right]\end{array}\right\}=0.$$

Therefore, we get

$$d^2(z,Qu) \le \hbar\max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+0-\emptyset(0).$$

Thus we get $z = \mathcal{Q}u$. Hence $z = \mathcal{Q}u = \mathcal{S}\mathcal{T}u$. Since $(\mathcal{Q}, \mathcal{S}\mathcal{T})$ are weakly compatible, so we have $\mathcal{Q}z = \mathcal{S}\mathcal{T}z$.Next, we will show that $\mathcal{T}z = z$. For this putting For this putting $x = x_{2n}$ and $y = \mathcal{T}u$ in (C_5), we get

$$[1 + \hbar d(\mathcal{AB}x_{2n}, \mathcal{STT}u)]d^2(\mathcal{P}x_{2n}, \mathcal{QT}u)$$

$$\leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n})d(\mathcal{STT}u, \mathcal{QT}u) \\ +d^2(\mathcal{STT}u, \mathcal{QT}u)d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n}) \end{bmatrix}, \\ d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n})d(\mathcal{AB}x_{2n}, \mathcal{QT}u)d(\mathcal{P}x_{2n}, \mathcal{STT}u), \\ d(\mathcal{AB}x_{2n}, \mathcal{QT}u)d(\mathcal{P}x_{2n}, \mathcal{STT}u)d(\mathcal{STT}u, \mathcal{QT}u) \end{Bmatrix}$$

$+\sigma(\mathcal{AB}x_{2n}, \mathcal{STT}u) - \emptyset\{\sigma(\mathcal{AB}x_{2n}, \mathcal{STT}u)\}$,where $\sigma(\mathcal{AB}x_{2n}, \mathcal{STT}u) =$

$$max \begin{Bmatrix} d^2(\mathcal{AB}x_{2n}, \mathcal{STT}u), \\ d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n})d(\mathcal{STT}u, \mathcal{QT}u), \\ d(\mathcal{AB}x_{2n}, \mathcal{QT}u)d(\mathcal{P}x_{2n}, \mathcal{STT}u), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}x_{2n}, \mathcal{P}x_{2n})d(\mathcal{AB}x_{2n}, \mathcal{QT}u) \\ +d(\mathcal{P}x_{2n}, \mathcal{STT}u)d(\mathcal{STT}z, \mathcal{QT}u) \end{bmatrix} \end{Bmatrix}$$

Taking limit as $n \to \infty$, we have and $\mathcal{QT} = \mathcal{TQ}$ gives $\mathcal{QT}u = \mathcal{TQ}u = \mathcal{T}z$ $(z = \mathcal{Q}u)$
$\mathcal{ST} = \mathcal{TS}$ imply $\mathcal{ST}(\mathcal{T}u) = \mathcal{TS}(\mathcal{T}u) = \mathcal{T}(\mathcal{ST}u) = \mathcal{T}z$.

$$[1 + \hbar d(z, \mathcal{T}z)]d^2(z, \mathcal{T}z)$$

$$\leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(z,z)d(\mathcal{T}z, \mathcal{T}z) \\ +d^2(\mathcal{T}z, \mathcal{T}z)d(z,z) \end{bmatrix}, \\ d(z,z)d(z, \mathcal{T}z)d(z, \mathcal{T}z), \\ d(z, \mathcal{T}z)d(z, \mathcal{T}z)d(\mathcal{T}z, \mathcal{T}z) \end{Bmatrix}$$

$+\sigma(z, \mathcal{T}z) - \emptyset\{\sigma(z, \mathcal{T}z)\}$,

where$\sigma(z, \mathcal{T}z) = max \begin{Bmatrix} d^2(z, \mathcal{T}z), \\ d(z,z)d(\mathcal{T}z, \mathcal{T}z), \\ d(z, \mathcal{T}z)d(z, \mathcal{T}z), \\ \frac{1}{2}\begin{bmatrix} d(z,z)d(z, \mathcal{T}z) \\ +d(z, \mathcal{T}z)d(\mathcal{T}z, \mathcal{T}z) \end{bmatrix} \end{Bmatrix} = d^2(z, \mathcal{T}z).$

Therefore, we get

$$[1 + \hbar d(z, \mathcal{T}z)]d^2(z, \mathcal{T}z) \leq \hbar max \begin{Bmatrix} \frac{1}{2}[0 + 0], \\ 0, \\ 0 \end{Bmatrix} + d^2(z, \mathcal{T}z) - \emptyset\left(d^2(z, \mathcal{T}z)\right).$$

i.e., $d^2(z, \mathcal{T}z) \leq 0$. Thus $z = \mathcal{T}z$.
Now $\mathcal{ST}z = \mathcal{T}z = z$imply $\mathcal{S}z = z$.
Hence $\mathcal{S}z = \mathcal{T}z = \mathcal{Q}z = z$.
As $\mathcal{Q}(\mathfrak{A}) \subseteq \mathcal{AB}(\mathfrak{A})$, there exists $w \in \mathfrak{A}$ such that $z = \mathcal{Q}z = \mathcal{AB}w$.
Next, we will show that $\mathcal{P}w = z$. For this putting $x = w$ and $y = x_{2n+1}$ in (C_5), we have

$$[1 + \hbar d(\mathcal{AB}w, \mathcal{ST}x_{2n+1})]d^2(\mathcal{P}w, \mathcal{Q}x_{2n+1})$$

$$\leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{AB}w, \mathcal{P}w)d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \\ +d^2(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1})d(\mathcal{AB}w, \mathcal{P}w) \end{bmatrix}, \\ d(\mathcal{AB}w, \mathcal{P}w)d(\mathcal{AB}w, \mathcal{Q}x_{2n+1})d(\mathcal{P}w, \mathcal{ST}x_{2n+1}), \\ d(\mathcal{AB}w, \mathcal{Q}x_{2n+1})d(\mathcal{P}w, \mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \end{Bmatrix}$$

$+\sigma(\mathcal{AB}w, \mathcal{ST}x_{2n+1}) - \emptyset\{\sigma(\mathcal{AB}w, \mathcal{ST}x_{2n+1})\}$,
where

$$\sigma(\mathcal{AB}w, \mathcal{ST}x_{2n+1}) = max \begin{Bmatrix} d^2(\mathcal{AB}w, \mathcal{ST}x_{2n+1}), \\ d(\mathcal{AB}w, \mathcal{P}w)d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}), \\ d(\mathcal{AB}w, \mathcal{Q}x_{2n+1})d(\mathcal{P}w, \mathcal{ST}x_{2n+1}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}w, \mathcal{P}w)d(\mathcal{AB}w, \mathcal{Q}x_{2n+1}) \\ +d(\mathcal{P}w, \mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1}, \mathcal{Q}x_{2n+1}) \end{bmatrix} \end{Bmatrix}$$

Taking limit as $n \to \infty$, we have

$$[1+\hbar d(z,z)]d^2(\mathcal{P}w,z)\le \hbar\, max\left\{\begin{matrix}\frac{1}{2}\begin{bmatrix}d^2(z,\mathcal{P}w)d(z,z)\\+d^2(z,z)d(z,\mathcal{P}w)\end{bmatrix},\\ d(z,\mathcal{P}w)d(z,z)d(\mathcal{P}w,z),\\ d(z,z)d(\mathcal{P}w,z)d(z,z)\end{matrix}\right\}$$

$+\sigma(z,z)-\emptyset\{\sigma(z,z)\}$,

$$\text{where}\sigma(z,z)=max\left\{\begin{matrix}d^2(z,z),\\ d(z,\mathcal{P}x)d(z,z),\\ d(z,z)d(\mathcal{P}w,z),\\ \frac{1}{2}\begin{bmatrix}d(z,\mathcal{P}w)d(z,z)\\+d(\mathcal{P}w,z)d(z,z)\end{bmatrix}\end{matrix}\right\}=0.$$

$$[1+\hbar d(z,z)]d^2(\mathcal{P}w,z)\le\ \hbar max\left\{\begin{matrix}\frac{1}{2}[0+0],\\ 0,\\ 0\end{matrix}\right\}+0-\emptyset(0).$$

i.e., $d^2(\mathcal{P}w,z)\le 0$. Thus $\mathcal{P}w=z$.
Since $(\mathcal{P},\mathcal{AB})$ are weakly compatible, so $\mathcal{P}$ and $\mathcal{AB}$ commute their coincidence point w, then we have $\mathcal{P}z=\mathcal{AB}z$.
Next, we will show that $\mathcal{P}z=z$. For this putting $x=z$ and $y=x_{2n+1}$ in (C_5), we have

$$[1+\hbar d(\mathcal{AB}z,\mathcal{ST}x_{2n+1})]d^2(\mathcal{P}z,\mathcal{Q}x_{2n+1})$$

$$\le \hbar\, max\left\{\begin{matrix}\frac{1}{2}\begin{bmatrix}d^2(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\\+d^2(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})d(\mathcal{AB}z,\mathcal{P}z)\end{bmatrix},\\ d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{AB}z,\mathcal{Q}x_{2n+1})d(\mathcal{P}z,\mathcal{ST}x_{2n+1}),\\ d(\mathcal{AB}z,\mathcal{Q}x_{2n+1})d(\mathcal{P}z,\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\end{matrix}\right\}$$

$+\sigma(\mathcal{AB}z,\mathcal{ST}x_{2n+1})-\emptyset\{\sigma(\mathcal{AB}z,\mathcal{ST}x_{2n+1})\}$,
where

$$\sigma(\mathcal{AB}z,\mathcal{ST}x_{2n+1})=max\left\{\begin{matrix}d^2(\mathcal{AB}z,\mathcal{ST}x_{2n+1}),\\ d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1}),\\ d(\mathcal{AB}z,\mathcal{Q}x_{2n+1})d(\mathcal{P}z,\mathcal{ST}x_{2n+1}),\\ \frac{1}{2}\begin{bmatrix}d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{AB}z,\mathcal{Q}x_{2n+1})\\+d(\mathcal{P}z,\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\end{bmatrix}\end{matrix}\right\}.$$

Taking limit as $n\to\infty$, we have

$$[1+\hbar d(\mathcal{P}z,z)]d^2(\mathcal{P}z,z)\le \hbar\, max\left\{\begin{matrix}\frac{1}{2}\begin{bmatrix}d^2(\mathcal{P}z,\mathcal{P}z)d(z,z)\\+d^2(z,z)d(\mathcal{P}z,\mathcal{P}z)\end{bmatrix},\\ d(\mathcal{P}z,\mathcal{P}z)d(\mathcal{P}z,z)d(\mathcal{P}z,z),\\ d(\mathcal{P}z,z)d(\mathcal{P}z,z)d(z,z)\end{matrix}\right\}$$

$+\sigma(\mathcal{P}z,z)-\emptyset\{\sigma(\mathcal{P}z,z)\}$,

$$\text{where}\sigma(\mathcal{P}z,z)=max\left\{\begin{matrix}d^2(\mathcal{P}z,z),\\ d(z,\mathcal{P}z)d(z,z),\\ d(\mathcal{P}z,z)d(\mathcal{P}z,z),\\ \frac{1}{2}\begin{bmatrix}d(\mathcal{P}z,\mathcal{P}z)d(\mathcal{P}z,z)\\+d(\mathcal{P}z,z)d(z,z)\end{bmatrix}\end{matrix}\right\}.$$

$$[1+\hbar d(\mathcal{P}z,z)]d^2(\mathcal{P}z,z)\le\ \hbar max\left\{\begin{matrix}\frac{1}{2}[0+0],\\ 0,\\ 0\end{matrix}\right\}+d^2(\mathcal{P}z,z)-\emptyset\big(d^2(\mathcal{P}z,z)\big).$$

i.e., $d^2(\mathcal{P}z,z)\le 0$. Thus $\mathcal{P}z=z$. Then $\mathcal{P}z=\mathcal{AB}z=z$.
Next, we will show that $\mathcal{B}z=z$. For this putting $x=\mathcal{B}z$ and $y=x_{2n+1}$ in (C_5), we have

$$[1+\hbar d(\mathcal{ABB}z,\mathcal{ST}x_{2n+1})]d^2(\mathcal{PB}z,\mathcal{Q}x_{2n+1})$$

$$\leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{ABB}z,\mathcal{PB}z)d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1}) \\ +d^2(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})d(\mathcal{ABB}z,\mathcal{PB}z)\end{bmatrix}, \\ d(\mathcal{ABB}z,\mathcal{PB}z)d(\mathcal{ABB}z,\mathcal{Q}x_{2n+1})d(\mathcal{PB}z,\mathcal{ST}x_{2n+1}), \\ d(\mathcal{ABB}z,\mathcal{Q}x_{2n+1})d(\mathcal{PB}z,\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1}) \end{Bmatrix}$$

$+\sigma(\mathcal{ABB}z,\mathcal{ST}x_{2n+1}) - \emptyset\{\sigma(\mathcal{ABB}z,\mathcal{ST}x_{2n+1})\}$,
where

$$\sigma(\mathcal{ABB}z,\mathcal{ST}x_{2n+1}) = max \begin{Bmatrix} d^2(\mathcal{ABB}z,\mathcal{ST}x_{2n+1}), \\ d(\mathcal{ABB}z,\mathcal{PB}z)d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1}), \\ d(\mathcal{ABB}z,\mathcal{Q}x_{2n+1})d(\mathcal{PB}z,\mathcal{ST}x_{2n+1}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{ABB}z,\mathcal{PB}z)d(\mathcal{ABB}z,\mathcal{Q}x_{2n+1}) \\ +d(\mathcal{PB}z,\mathcal{ST}x_{2n+1})d(\mathcal{ST}x_{2n+1},\mathcal{Q}x_{2n+1})\end{bmatrix} \end{Bmatrix}.$$

Taking limit as $n \to \infty$ and $\mathcal{PB} = \mathcal{BP}$, $\mathcal{AB} = \mathcal{BA}$ implies $\mathcal{P}(\mathcal{B}z) = \mathcal{B}(\mathcal{P}z) = \mathcal{B}z$ and $\mathcal{AB}(\mathcal{B}z) = \mathcal{B}(\mathcal{AB})z = \mathcal{B}z$

$$[1 + \hbar d(\mathcal{B}z,z)]d^2(\mathcal{B}z,z) \leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{B}z,\mathcal{B}z)d(z,z) \\ +d^2(z,z)d(\mathcal{B}z,\mathcal{B}z)\end{bmatrix}, \\ d(\mathcal{B}z,\mathcal{B}z)d(\mathcal{B}z,z)d(\mathcal{B}z,z), \\ d(\mathcal{B}z,z)d(\mathcal{B}z,z)d(z,z) \end{Bmatrix}$$

$+\sigma(\mathcal{B}z,z) - \emptyset\{\sigma(\mathcal{B}z,z)\}$,

where $\sigma(\mathcal{B}z,z) = max \begin{Bmatrix} d^2(\mathcal{B}z,z), \\ d(\mathcal{B}z,\mathcal{B}z)d(z,z), \\ d(\mathcal{B}z,z)d(\mathcal{B}z,z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{B}z,\mathcal{B}z)d(\mathcal{B}z,z) \\ +d(\mathcal{B}z,z)d(z,z)\end{bmatrix} \end{Bmatrix} = d^2(\mathcal{B}z,z).$

$$[1 + \hbar d(\mathcal{B}z,z)]d^2(\mathcal{B}z,z) \leq \hbar max \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix} + d^2(\mathcal{B}z,z) - \emptyset\left(d^2(\mathcal{B}z,z)\right).$$

i.e., $d^2(\mathcal{B}z,z) \leq 0$. Thus $\mathcal{B}z = z$. Then $z = \mathcal{AB}z = \mathcal{A}z$. Therefore $z = \mathcal{B}z = \mathcal{A}z = \mathcal{P}z$.
Hence in all we have $z = \mathcal{B}z = \mathcal{A}z = \mathcal{P}z = \mathcal{S}z = \mathcal{T}z = \mathcal{Q}z$.
Case II. When $\mathcal{P}(\mathfrak{U})$ is complete follows from above case as $\mathcal{P}(\mathfrak{U}) \subseteq \mathcal{ST}(\mathfrak{U})$.
Case III. When $\mathcal{AB}(\mathfrak{U})$ is complete. This case follows by symmetry. As $\mathcal{Q}(\mathfrak{U}) \subseteq \mathcal{AB}(\mathfrak{U})$, therefore the result also holds when $\mathcal{Q}(\mathfrak{U})$ is complete.
Uniqueness. Let m be another common fixed point of $\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T},\mathcal{P}$ and $\mathcal{Q}$. Then $m = \mathcal{B}m = \mathcal{A}m = \mathcal{P}m = \mathcal{S}m = \mathcal{T}m = \mathcal{Q}m$.
Finally, we will show that $z = m$. For this putting $x = z$ and $y = m$ in (C_5), we have

$$[1 + \hbar d(\mathcal{AB}z,\mathcal{ST}m)]d^2(\mathcal{P}z,\mathcal{Q}m)$$

$$\leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{ST}m,\mathcal{Q}m) \\ +d^2(\mathcal{ST}m,\mathcal{Q}m)d(\mathcal{AB}z,\mathcal{P}z)\end{bmatrix}, \\ d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{AB}z,\mathcal{Q}m)d(\mathcal{P}z,\mathcal{ST}m), \\ d(\mathcal{AB}z,\mathcal{Q}m)d(\mathcal{P}z,\mathcal{ST}m)d(\mathcal{ST}m,\mathcal{Q}m) \end{Bmatrix}$$

$+\sigma(\mathcal{AB}z,\mathcal{ST}m) - \emptyset\{\sigma(\mathcal{AB}z,\mathcal{ST}m)\}$,

where $\sigma(\mathcal{AB}z,\mathcal{ST}m) = max \begin{Bmatrix} d^2(\mathcal{AB}z,\mathcal{ST}m), \\ d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{ST}m,\mathcal{Q}m), \\ d(\mathcal{AB}z,\mathcal{Q}m)d(\mathcal{P}z,\mathcal{ST}m), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}z,\mathcal{P}z)d(\mathcal{AB}z,\mathcal{Q}m) \\ +d(\mathcal{P}z,\mathcal{ST}m)d(\mathcal{ST}m,\mathcal{Q}m)\end{bmatrix} \end{Bmatrix}$

$$[1 + \hbar d(z,m)]d^2(z,m) \leq \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(z,z)d(m,m) \\ +d^2(m,m)d(z,z)\end{bmatrix}, \\ d(z,z)d(z,m)d(z,m), \\ d(z,m)d(z,m)d(m,m) \end{Bmatrix}$$

$+\sigma(z,m) - \emptyset\{\sigma(z,m)\}$,

$$\text{where}\ \sigma(z,m) = max \left\{ \begin{matrix} d^2(z,m), \\ d(z,z)d(m,m), \\ d(z,m)d(z,m), \\ \frac{1}{2}\begin{bmatrix} d(z,z)d(z,m) \\ +d(z,m)d(m,m) \end{bmatrix} \end{matrix} \right\} = d^2(z,m).$$

$$[1+\hbar d(z,m)]d^2(z,m) \le \hbar max \left\{ \begin{matrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{matrix} \right\} + d^2(z,m) - \emptyset\big(d^2(z,m)\big).$$

i.e., $d^2(z,m) \le 0$. Thus $z = m$. Hence z be a unique common fixed point of $\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T},\mathcal{P}$ and $\mathcal{Q}$.

Corollary2.1 Let $\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T},\mathcal{P}$ and $\mathcal{Q}$ be six self-mappings of metric space $(\mathfrak{A},d)$ satisfying (C_1),(C_2),(C_3),(C_4) and the following condition:

(C_6) $d^2(\mathcal{P}x,\mathcal{Q}y) \le \sigma(\mathcal{AB}x,\mathcal{ST}y) - \emptyset\{\sigma(\mathcal{AB}x,\mathcal{ST}y)\}$

$$\text{where}\ \sigma(\mathcal{AB}x,\mathcal{ST}y) = max \left\{ \begin{matrix} d^2(\mathcal{AB}x,\mathcal{ST}y), \\ d(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{ST}y,\mathcal{Q}y), \\ d(\mathcal{AB}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{ST}y), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AB}x,\mathcal{P}x)d(\mathcal{AB}x,\mathcal{Q}y) \\ +d(\mathcal{P}x,\mathcal{ST}y)d(\mathcal{ST}y,\mathcal{Q}y) \end{bmatrix} \end{matrix} \right\}.$$

for all $x, y \in \mathfrak{A}$and $\emptyset$: $[0,\infty) \to [0,\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > t$, for each $t > 0$.Then $\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T},\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathfrak{A}$.

Proof.$\hbar$ = 0 in Theorem 2.1, we have the result.

In the Theorem 2.1, if we take $\mathcal{T} = \mathcal{B} = \mathcal{Q} = I$ (Identity mapping), the we have the following corollary

Corollary 2.2 Let $\mathcal{A},\mathcal{S},\mathcal{P}$ be three self-mappings of metric space $(\mathfrak{A},d)$ satisfying (C_1),(C_3),(C_4) and the following condition:

$$[1+\hbar d(\mathcal{A}x,\mathcal{S}y)]d^2(\mathcal{P}x,\mathcal{Q}y) \le \hbar\ max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{A}x,\mathcal{P}x)d(\mathcal{S}y,y) \\ +d^2(\mathcal{S}y,y)d(\mathcal{A}x,\mathcal{P}x) \end{bmatrix}, \\ d(\mathcal{A}x,\mathcal{P}x)d(\mathcal{A}x,y)d(\mathcal{P}x,\mathcal{S}y), \\ (\mathcal{AB}x,y)d(\mathcal{P}x,\mathcal{S}y)d(\mathcal{S}y,y) \end{matrix} \right\}$$

$+\sigma(\mathcal{A}x,\mathcal{S}y) - \emptyset\{\sigma(\mathcal{A}x,\mathcal{S}y)\}$,

$$\text{where}\ \sigma(\mathcal{A}x,\mathcal{S}y) = max \left\{ \begin{matrix} d^2(\mathcal{A}x,\mathcal{S}y), \\ d(\mathcal{A}x,\mathcal{P}x)d(\mathcal{S}y,y), \\ d(\mathcal{A}x,y)d(\mathcal{P}x,\mathcal{S}y), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}x,\mathcal{P}x)d(\mathcal{A}x,y) \\ +d(\mathcal{P}x,\mathcal{S}y)d(\mathcal{S}y,y) \end{bmatrix} \end{matrix} \right\}$$

for all $x, y \in \mathfrak{A}$, $\hbar \ge 0$ is a real number and $\emptyset$: $[0,\infty) \to [0,\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$, for each $t > 0$.Then $\mathcal{A},\mathcal{S},\mathcal{P}$have a unique common fixed point in $\mathfrak{A}$.

In the Theorem 2.1, if we take $\mathcal{T} = \mathcal{B} = I$ (Identity mapping), the we have the following corollary

Corollary2.3 Let $\mathcal{A},\mathcal{S},\mathcal{P}$ and $\mathcal{Q}$ be four self-mappings of metric space $(\mathfrak{A},d)$ satisfying (C_1),(C_2), (C_3), (C_4) and the following condition:

$$[1+\hbar d(\mathcal{A}x,\mathcal{S}y)]d^2(\mathcal{P}x,\mathcal{Q}y)$$

$$\le \hbar\ max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(\mathcal{A}x,\mathcal{P}x)d(\mathcal{S}y,\mathcal{Q}y) \\ +d^2(\mathcal{S}y,\mathcal{Q}y)d(\mathcal{A}x,\mathcal{P}x) \end{bmatrix}, \\ d(\mathcal{A}x,\mathcal{P}x)d(\mathcal{A}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{S}y), \\ d(\mathcal{A}x,\mathcal{Q}y)d(\mathcal{P}x,\mathcal{S}y)d(\mathcal{S}y,\mathcal{Q}y) \end{matrix} \right\} + \sigma(\mathcal{A}x,\mathcal{S}y) - \emptyset\{\sigma(\mathcal{A}x,\mathcal{S}y)\},$$

where $\sigma(\mathcal{A}x, \mathcal{S}y) = max \left\{ \begin{array}{c} d^2(\mathcal{A}x, \mathcal{S}y), \\ d(\mathcal{A}x, \mathcal{P}x)d(\mathcal{S}y, \mathcal{Q}y), \\ d(\mathcal{A}x, \mathcal{Q}y)d(\mathcal{P}x, \mathcal{S}y), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{A}x, \mathcal{P}x)d(\mathcal{A}x, \mathcal{Q}y) \\ +d(\mathcal{P}x, \mathcal{S}y)d(\mathcal{S}y, \mathcal{Q}y) \end{array} \right] \end{array} \right\}$,

for all $x, y \in \mathfrak{A}$, $h \geq 0$ is a real number and $\emptyset$: $[0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > t$, for each $t > 0$. Then $\mathcal{A}, \mathcal{S}, \mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathfrak{A}$.

Proof. Taking $\mathcal{T} = \mathcal{B} = I$ in Theorem 2.1, we have the result.

Example 2.1 Let $X = [0,1]$ and d be a usual metric. Define the self mappings $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{P}$ and $\mathcal{Q}$ on X by $\mathcal{P}x = \mathcal{Q}x = x/3$ and $\mathcal{A}x = \mathcal{B}x = \mathcal{S}x = \mathcal{T}x = 3x/4$ and $\emptyset$: $[0, \infty) \rightarrow [0, \infty)$ is a continuous function defined by $\emptyset(t) = t/3$ which satisfied $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Taking $< x_n > = < \frac{1}{n} >$, it is clear that pairs $(\mathcal{P}, \mathcal{AB})$ and $(\mathcal{Q}, \mathcal{ST})$ are weakly compatible. Therefore, all the condition of Theorem 2.1 are satisfied, then we can obtain $\mathcal{S}0 = \mathcal{T}0 = \mathcal{A}0 = \mathcal{B}0 = \mathcal{P}0 = \mathcal{Q}0 = 0$, so $x = 0$ is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{P}$ and $\mathcal{Q}$. In fact, $x = 0$ is the unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{P}$ and $\mathcal{Q}$.

Conclusion:

In this paper, we prove a common fixed point theorem for six self mapping using weakly compatible mapping in a metric space. At the last we give corollaries and example in support of our theorem.

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